Alma MATH Challenge; November 2018

Include clear and complete justification for your solutions to these problems. You may use standard results of undergraduate-level mathematics courses. Identify any results you use and be sure to verify the hypotheses. Partial credit may be awarded for an incomplete proof or a reasonable idea of how to solve the problem. You may use a hand-held calculator provided it does not have symbolic manipulation capability. External aid from books, notes, calculators with symbolic algebra system, computers, smart devices, etc. is prohibited. The problems are listed in increased difficulty (according to the author subjective perspective, of course). The test is heavily based on calculations and just a few ideas are needed.

1. On the unit segment $[0,1]$ we choose two points at random (uniform distribution). These points divide $[0,1]$ into three segments, of lengths a, b and c. Show that the mean of the random variable $X = max(a,b,c)$ is

$$E(X) = \frac{11}{18}.$$

2. (i) Consider the polynomials $C_0(x) \equiv 1$, $C_1(x) = x$ and $C_2(x) = 2x^2 - 1$. For every k and l, $0 \leq k \leq l \leq 2$, calculate the improper integrals

$$\int_{-1}^{1} \frac{C_k(x)C_l(x)}{\sqrt{1-x^2}} dx.$$

(ii) If a continuous function $f : [-1,1] \to \mathbb{R}$ has the property that $\int_{-1}^{1} \frac{x^k f(x)}{\sqrt{1-x^2}} dx = \pi$ for $k \in \{0,1,2\}$, show that

$$\int_{-1}^{1} \frac{f(x)^2}{\sqrt{1-x^2}} dx \geq 5\pi.$$

(iii) Generalize part (ii)!

3. The Fibonacci sequence $\{F_n\}$, is defined by $F_0 = 0$, $F_1 = 1$, and the recurrence $F_k = F_{k-1} + F_{k-2}$ for $k \geq 2$. For $n \geq 1$, prove that if

$$G_n := \frac{F_{3n}}{2F_n}$$

is an integer, then G_n is congruent to 0 or 1 modulo 3.

4. We use the usual notation, $[XYZ...]$, for the area of the polygon $XYZ...$. Let ABCD be a convex quadrilateral and O be the intersecting point of its diagonals. Let K, L be points on the side $\overline{CD}$ such that $\overline{OK}||\overline{AD}$ and $\overline{OL}||\overline{BC}$. Prove that $DK^2 + LC^2 = KL^2$ is equivalent to

$$\frac{[ABCD]}{[OCD]} = 2.$$

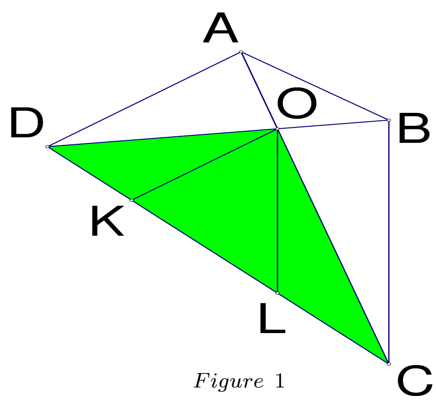

Figure 1

5. Two square matrices $A, B \in M_n(\mathbb{Q})$ satisfy $A^2 - 2B^2 = AB - BA$. Show that

$$det(AB - BA) = 0.$$

6. We let G be a finite group of order $n \in \mathbb{N}$. For every $m \in \{1, 2, ..., n-1\}$ such that $\gcd(m, n) = 1$ define the map $f_m : G \to G$ by $f_m(x) = x^m$ for $x \in G$.

(i) Prove that f_m is a bijection.

(ii) Show that the set $H := \{f_m | m \in U(n)\}$ together with the composition of functions is a group isomorphic to $\widehat{H} := U(n)/K$, where $U(n)$ is the set of all integers m, $0 < m < n$ that are relatively prime to n, $K := \{p \in U(n) | p \equiv 1 \ (mod\ M)\}$ with $M = lcm\{|x| | x \in G\}$ ($|x|$ denotes the order of the element x in G).

7. Suppose that H is a finite subgroup of the group G of all continuous bijections of $[0,1]$ to itself. Show that H must have order 1 or 2, and there are uncountably many such subgroups.

8. For $n \in \mathbb{N}$, consider in $\mathbb{R}^3$ the regular tetrahedron with vertices $O(0,0,0)$, $A(4n,n,n)$, $B(n,4n,n)$ and $C(n,n,4n)$. Show that the number N of points (x,y,z), $(x,y,z \in \mathbb{Z})$ inside or on the boundary of the tetrahedron $OABC$ is given by

$$N = \frac{1}{2}(6n+1)(3n^2+n+2).$$

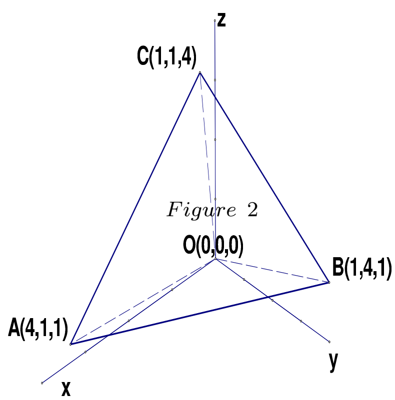

Figure 2

9. A hound dog is at location D (see Figure 3) and a rabbit is at location R, having a distance between them of $DR = d$. The dog and the rabbit run with constant speeds v_h and v_r respectively and $q := \frac{v_h}{v_r} > 1$. At a given point in time the rabbit sees the dog and starts running in a straight path in the direction of point C (see Figure 3), with $m(\angle CRD) = \alpha \in (0, \pi)$. The dog is chasing the rabbit, i.e., its velocity is directed toward the rabbit at any time, and it catches the rabbit at point C with $RC = d'$.

(a) Show that if $\alpha = 90^\circ$ and $d' = d$, then $q = \frac{1+\sqrt{5}}{2}$ (the Golden Ratio).

(b) Prove that $d' = d\left(\frac{q-\cos\alpha}{q^2-1}\right)$.

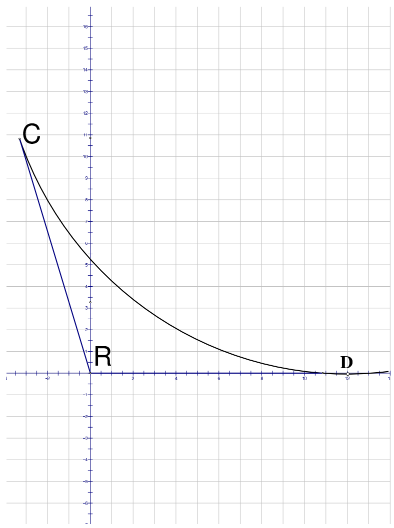

Figure 3

10. Suppose $0 < x_0 < 1$, and for $n \geq 1$ define

$$x_n = \frac{1}{n}\sum_{j=0}^{n-1}(x_j - x_j^2). \qquad (1)$$

Find $\ell := \lim_{n\to\infty} x_n \ln n$.