Michigan Autumn Take-Home (MATH) Challenge – November 4, 2017

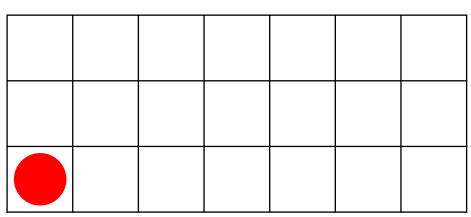
1. Find, with proof, all prime numbers p and q where both $p+q$ and $p-q$ are also prime numbers.

2. Suppose n is an odd integer.
   (a) Show that n^2+23 has at least six distinct divisors.
   (b) Find the smallest integer k so that n^2+k always has at least six divisors, and prove you've found the smallest such k.

3. Let A, B, and C be three points in the plane. Is there always a square whose sides contain these three points? Prove your answer.

4. Fix a positive integer k, and consider the set S_k of all possible sums of the form $\pm 1 \pm 2 \cdots \pm k$. For example, $S_2 = \{-3, -1, 1, 3\}$.
   (a) Show that, for any integer n (not assumed to be positive!), it is possible to find a k so that $n \in S_k$.
   (b) What is the smallest k with $2017 \in S_k$?
   (c) Is there a largest value of k with $2017 \in S_k$? If so, find it; if not, explain why no such maximum exists.

5. A game is played with one checker on a $3 \times n$ checkerboard, where $n > 1$ is a positive integer. The checker is initially in the lower left square, as in the picture, which shows the 3×7 grid. Alice and Betty alternate moving the checker one square at a time. On any move, the checker may move exactly one square up, down, or right, but not left. Further, the checker can never occupy the same square twice. The winner is the person who moves the checker into the last column for the first time.



For instance, the sequence of moves $RRUURDR$ is legal, but RUD is not (same square occupied twice) and UUU is not (the checker would leave the board). For all $n > 1$, determine which player has a winning strategy. What is that strategy?

6. Let $f(x) = \dfrac{4x}{4x^2 - 1}$. Find $f^{(2017)}(0)$, the 2017^{th} derivative of $f(x)$, evaluated at $x = 0$.

7. You are doodling during class, and you decide to randomly place the integers from 1 to 2017 (inclusive) around a circle. Then you draw a line segment between two of these integers if they differ by a power of 2.
   (a) How many line segments are there in your drawing?
   (b) The *degree* of an integer is the number of line segments that have it as an endpoint. What is the maximum degree of any integer in your drawing? How many different integers have that maximum degree?

8. Let n be a positive integer. Show that it is always possible to find (not necessarily distinct) prime numbers $p_1, p_2, \ldots, p_m$ and $q_1, q_2, \ldots, q_k$ so that

$$n = \frac{p_1!p_2!\cdots p_m!}{q_1!q_2!\cdots q_k!}$$

9. Given a positive rational number $r = \frac{a}{b} < 1$ written in lowest possible terms, you may transform r by adding 1 to either the numerator or the denominator of r, then reducing the result to lowest possible terms. You are allowed to iterate this transformation as many times as you like, but your fractions must stay less than 1. For example, here is a sequence of steps that transforms $\frac{5}{8}$ to $\frac{2}{3}$.

$$\frac{5}{8} \to \frac{6}{8} = \frac{3}{4} \to \frac{3}{5} \to \frac{3}{6} = \frac{1}{2} \to \frac{1}{3} \to \frac{2}{3}$$

   (a) Show how to transform $\frac{8}{13}$ to $\frac{5}{6}$.
   (b) Let $\frac{a}{b}$ be a rational number with $0 < a < b$ written in lowest terms. Show that it is always possible to transform $\frac{a}{b}$ to $\frac{1}{2}$ in a finite number of steps.
   (c) (Extra Credit!) Let r_1 and r_2 be two distinct rationals between 0 and 1. Is it always possible to find a sequence of steps as above that will transform r_1 to r_2? If so, describe such a procedure; if not, give a specific example where this fails.

10. A circle of radius 10 is placed so that it is tangent to both arms of the parabola $y = x^2$. We then place a second circle beneath the first circle, with this second circle as large as possible (so it will be tangent to the two arms of the parabola and the first circle). Continue in this way, adding new circles below the tower of circles already drawn. (The figure to the right shows the tower after 3 circles have been added.) Stop when you can't add any more circles. How many circles will there be in your tower? What is the radius of the smallest circle in the tower?