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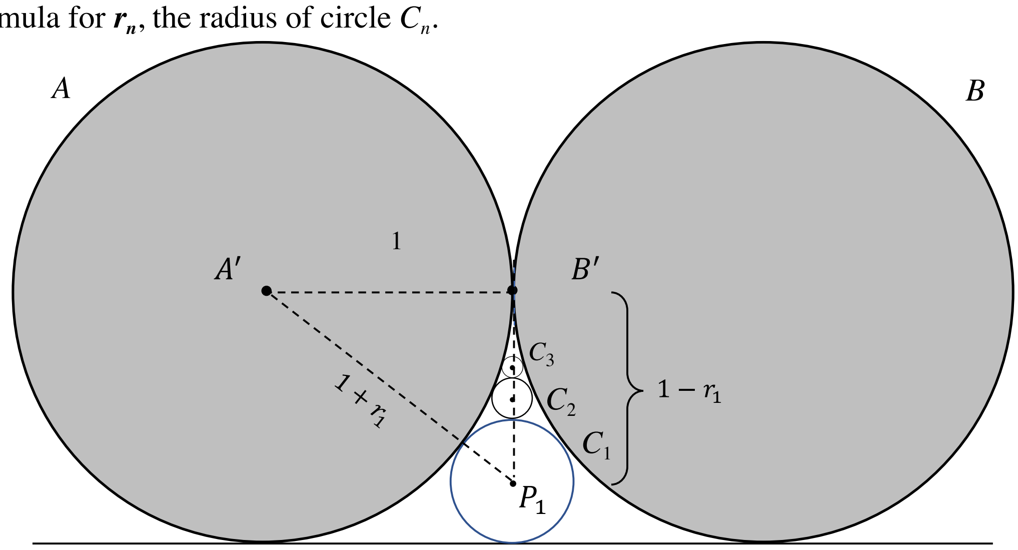
Solution to January's
ALMA COLLEGE MATH CHALLENGE

Two circles A and B, each with a radius of 1, are tangent to each other and to a horizontal line. Circle C_1, with a radius of r_1, is drawn so that it's tangent to circle A, circle B, and the horizontal line. Circle C_2, with a radius of r_2, is drawn so that it's tangent to circles A, B, and C_1. Continuing in this manner, circle C_n, with a radius of r_n, is drawn so that it's tangent to circles A, B, and C_{n-1}.
Find a formula for $\boldsymbol{r_n}$, the radius of circle C_n.



Solution by ***Dimash Umirbayev****, Ann Arbor Huron High School*

Let A' be the center of circle A and let B' be the point at which circles A and B are tangent. Lastly, let $P_1, P_2,$ and P_3 be the centers of circles $C_1, C_2,$ and C_3, respectively.

Triangle $B'A'P_1$ is a right triangle with legs of lengths 1 and $1 - r_1$ and hypotenuse $1 + r_1$. Using the Pythagorean Theorem, we have

$$\begin{aligned} 1^2 + (1 - r_1)^2 &= (1 + r_1)^2. \\ 1 + 1 - 2r_1 + r_1^2 &= 1 + 2r_1 + r_1^2 \\ 4r_1 &= 1 \\ r_1 &= \tfrac{1}{4}. \end{aligned}$$

Triangle $B'A'P_2$ is a right triangle with legs of lengths 1 and $\frac{1}{2} - r_2$ and hypotenuse $1 + r_2$. Using the Pythagorean Theorem, we have

$$\begin{aligned} 1^2 + \left(\tfrac{1}{2} - r_2\right)^2 &= (1 + r_2)^2. \\ 1 + \tfrac{1}{4} - r_2 + r_2^2 &= 1 + 2r_2 + r_2^2 \\ 3r_2 &= \tfrac{1}{4} \\ r_2 &= \tfrac{1}{12}. \end{aligned}$$

Triangle $B'A'P_3$ is a right triangle with legs of lengths 1 and $\frac{1}{3} - r_3$ and hypotenuse $1 + r_3$. Using the Pythagorean Theorem, we have

$$\begin{aligned} 1^2 + \left(\tfrac{1}{3} - r_3\right)^2 &= (1 + r_3)^2. \\ 1 + \tfrac{1}{9} - \tfrac{2}{3}r_3 + r_3^2 &= 1 + 2r_3 + r_3^2 \\ \tfrac{8}{3}r_3 &= \tfrac{1}{9} \\ r_3 &= \tfrac{1}{24}. \end{aligned}$$

Following this pattern, triangle $B'A'P_n$ is a right triangle with legs of lengths 1 and $\frac{1}{n} - r_n$ and hypotenuse $1 + r_n$. Thus,

$$\begin{aligned} 1^2 + \left(\tfrac{1}{n} - r_n\right)^2 &= (1 + r_n)^2. \\ 1 + \tfrac{1}{n^2} - \tfrac{2}{n}r_n + r_n^2 &= 1 + 2r_n + r_n^2 \\ \tfrac{1}{n^2} - \tfrac{2}{n}r_n &= 2r_n \\ 1 - 2nr_n &= 2n^2 r_n \\ 1 &= (2n^2 + 2n)r_n \\ r_n &= \tfrac{1}{2n(n+1)}. \ \blacksquare \end{aligned}$$

Also solved by: **Jeongin Bae**, **Allison Baker**, **Mary Bashshur**, **Rayaan Brar**, **Jerry Li**, **Daniel Muenz**, **Joseph Oh**, **Miles Peters**, **Joann Sun** (Ann Arbor Huron H.S.); **Alec Kussro**, **Julia Young** (Fenton H.S.); **Auden Chase** (Holt H.S.); **Aanchal Jain** (Okemos H.S.); **Mikhail Fedoseev** (South Lyon H.S.).

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