2016 Michigan Autumn Take-Home Challenge

Provide clear and complete justification for your solutions to these problems. You may use standard results of undergraduate-level mathematics courses. But identify any results you use and be sure to verify the hypotheses. Partial credit may be awarded for an incomplete proof or a reasonable idea of how to solve the problem. You may use a hand-held calculator provided it does not have symbolic manipulation capability. External aid from books, notes, calculators with symbolic algebra system, computers, smart devices, etc. is prohibited.

1. Show that a triangle with sides of lengths 4, 5, and 6 has one of its angles that is twice another.

2. Find $\displaystyle\lim_{x\to 0}\left(\frac{1+2^x+2^{2x}}{3}\right)^{1/x}$.

3. Consider the integers $U_k = \displaystyle\sum_{i=1}^{k} 10^{i-1}$. That is, U_k written in ordinary base-ten notation consists of the digit 1 repeated k times. Prove for any positive integer n relatively prime to 10, there is a value of k such that n divides U_k. What can you say in general about the smallest value of k (in terms of n) for which n divides U_k?

4. This problem makes use of the basic fact from complex analysis that $e^z = \displaystyle\sum_{k=0}^{\infty}\frac{z^n}{n!}$ for all complex numbers z. Thus, for example,

$$e = e^1 = \sum_{n=0}^{\infty}\frac{1}{n!} = 1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots .$$

   a. Find a closed-form expression for $\displaystyle\sum_{n=0}^{\infty}\frac{1}{(2n)!}$.

   b. Find a closed-form expression for $\displaystyle\sum_{n=0}^{\infty}\frac{1}{(3n)!}$.

5. For a positive integer n and a real number $r > 0$, the n-dimensional ball of radius r in $\mathbb{R}^n$ is the set

$$B^n(r) = \{(x_1,\ldots,x_n)\in\mathbb{R}^n \mid x_1^2+\cdots+x_n^2 \le r^2\}.$$

   We know that the length of $B^1(r)$ is $2r$, the area of $B^2(r)$ is πr^2, and the volume of $B^3(r)$ is $\frac{4}{3}\pi r^3$. The n-dimensional measure for geometric objects in $\mathbb{R}^n$ works just like length, area, and volume in $\mathbb{R}^1$, $\mathbb{R}^2$, and $\mathbb{R}^3$. For example, the 4-dimensional measure of the box

$$\left\{(x_1,x_2,x_3,x_4)\in\mathbb{R}^4 \;\middle|\; |x_i|\le 1 \text{ for } i=1,2,3,4\right\}$$

   is $2\times 2\times 2\times 2 = 16$.

   a. Find the n-dimensional measure $\mu_n(r)$ of $B^n(r)$. Suggestion: warm up by reviewing the two techniques from your calculus course for computing the volume of $B^3(r)$ based on the measures of objects of lower dimensions.

   b. Show that $\displaystyle\lim_{n\to\infty}\mu_n(1) = 0$.

   c. In which dimension n is $\mu_n(1)$ largest?

6. Let A be a set of $2n$ points in the plane, no three of which are collinear. Suppose n of these points are colored red and the other n are colored green. Prove or disprove: There are n closed straight line segments, no two with a point in common, such that the endpoints are points of A having different colors.

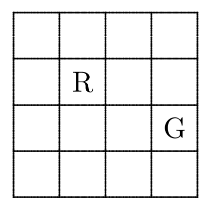
7. The fundamental notion of equivalence in topology is defined as follows: Two geometric objects are *homeomorphic* if and only if there is a continuous bijection between the two objects such that the inverse in also continuous. This bijection (known as a *homeomorphism*) is often viewed as a continuous deformation of one geometric object into another. Topologists typically give a homeomorphism by describing the bijection between the two objects in such a way that it is geometrically clear that points close together in one object correspond to points close together in the other object. For example, a circle and a square are homeomorphic since we can center the circle and square at the origin in a plane and radially project the points of the square onto corresponding points of the circle. This description makes it clear (even without an analytic formula) that the correspondence is one-to-one, onto, continuous, and has a continuous inverse.

   Let L be the set of lines in the plane. Two nonvertical line in L are close if their slopes and y-intercepts are close. To include vertical lines, consider the equation of a line as defining x as a function of y. Call the slope of this inverse function the *inverse slope* of the line. For example, the line $y = 2x + 5$ can be written $x = \frac{1}{2}y - \frac{5}{2}$, so its slope is 2 and its inverse slope is $\frac{1}{2}$. And a vertical line $x = c$ has inverse slope 0. Now two nonhorizontal lines are close if their inverse slopes and x-intercepts are close.

   a. Let S be the subset of L consisting of the lines through the origin. The space S is homeomorphic to what familiar geometric object? Describe the bijection between S and this geometric object carefully enough that is geometrically clear that the two objects are homeomorphic.

   b. The entire space L is homeomorphic to what familiar geometric object? Again, describe the bijection between L and this geometric object carefully enough that is geometrically clear that the two objects are homeomorphic.

8. You are playing a solitaire game with a red marker and a green marker on a 4×4 chessboard. On each turn, you can move one of the markers to a different unoccupied square. You win by realizing all $16 \times 15 = 240$ board positions without repeating a position.



   a. First suppose that a marker may jump to any unoccupied position, and the pieces must be moved alternately (red, then green, then red, then green, and so forth). Is it possible to win this game? Give a winning strategy or explain why winning is impossible.

   b. Now suppose that you do not have to move the pieces alternately: on each move you can move whichever marker you choose. However, now a marker can only move to a square that is adjacent (forward, backward, left, or right, not diagonally). Is it possible to win this game? Give a winning strategy or explain why winning is impossible.

9. a. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function with $f'(x) > 0$ for all $x \in \mathbb{R}$. Use the Mean Value Theorem of calculus to prove that f is strictly increasing. That is, if $x_1 < x_2$, then $f(x_1) < f(x_2)$.

   b. What about a pointwise version of this result? Suppose a differentiable function $f : \mathbb{R} \to \mathbb{R}$ satisfies $f'(0) > 0$. Prove there is an interval (a, b) with $a < 0 < b$ such that f is increasing on (a, b), or give a counterexample for which there is no such interval on which the function is increasing.

10. a. Show that $\sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + \cdots}}}}$ converges to a value less than or equal to 3.

   Suggestion: Let $s_n = \sqrt{1 + 2\sqrt{1 + \cdots + (n-1)\sqrt{1 + n}}}$ be the partial expression that stops with the term n added under $n - 1$ nested square roots. Look for patterns in the numerical computation of s_n for modest values of n.

   b. Show that the expression is equal to 3.