Alma MATH Challenge; November 2015

Correct answers alone are not worth many points. Full details of why the answer is correct are required, except in #1 and #8; for those, the answer is sufficient. Two hand-in sheets with blank tables are included for these two problems.

1. Serious Implications

Complete this logical crossword (use extra sheet attached) by placing either T (True) or F (False) into each square. Notation: $\neg$ means "not"; $\vee$ means "or"; $\Rightarrow$ means "implies". Each of the eight clues is an equation that refers to the entire row or column, viewed as a list. For example, the first clue indicates that the vector for (a) is the negation of the vector for (f), which would hold for, say, (a) being $TFFT$ and (f) being $FTTF$.

	e	f	g	h
a				
b				
c				
d				

ACROSS		DOWN	
a.	$\neg$ **f**	**e.**	**e** $\Rightarrow$ **b**
b.	**b** $\vee$ **b**	**f.**	$\neg$ **c**
c.	**b** $\Rightarrow$ **c**	**g.**	$\neg$ **(g** $\Rightarrow$ **(**$\neg$ **g))**
d.	**a** $\vee$ $\neg$ **h**	**h.**	$\neg\neg$ **h**

2. Four-Way Tennis

Alice, Bob, and Charlie play tennis in sets. Two of them (chosen randomly) play a set and the winner stays on the court for the next set, with the loser replaced by the idle player. At the end of the day, Alice played 15 sets, Bob played 14 sets, and Charlie played 9 sets. Who played in the 13th set?

3. How Much is a Penny Worth?

Alice tosses 99 fair coins and Bob tosses 100. What is the probability that Bob gets more (meaning: strictly more) heads than Alice?

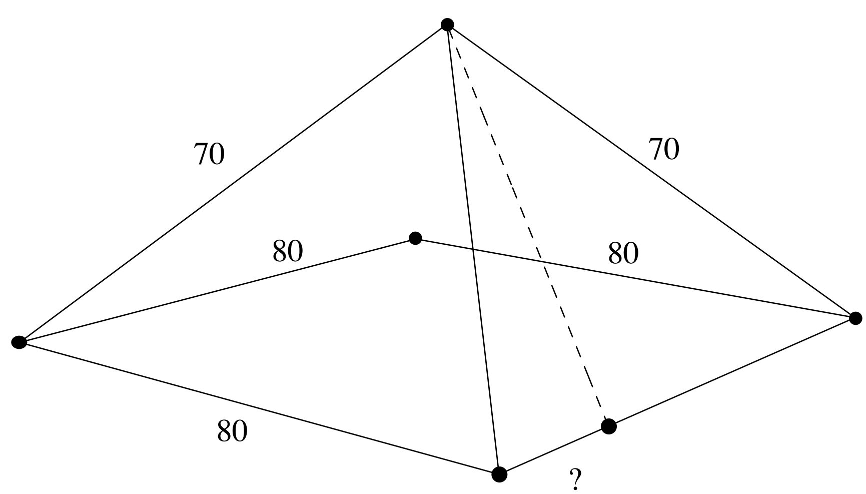
4. Football is a Zero-Sum-Square Game

The Michigan Intercollegiate Athletic Association football standings in the Fall of 2015 are shown below, where W denotes Wins and L denotes Losses. Football is a zero-sum game, so $\Sigma W = \Sigma L$ (21 in the table). But we see also that $\Sigma W^2 = \Sigma L^2$ (79 in the table). Prove that, in any tournament with no ties where each team plays each other team once, this relation holds: $\Sigma W^2 = \Sigma L^2$.

	W	L	W^2	L^2
Team 1	5	1	25	1
Team 2	4	2	16	4
Team 3	4	2	16	4
Team 4	3	3	9	9
Team 5	3	3	9	9
Team 6	2	4	4	16
Team 7	0	6	0	36
sums	21	21	79	79

5. Climb the Pyramid

You're standing next to a pyramid with a square base, 80 feet on a side. The distance from each corner of the base to the top is 70 feet. You want to start at some point along the square base and walk up a face in a straight line to the top of the pyramid (dashed line in figure) so that your path makes a 45° angle with the ground. How far from a corner of the base should you start?



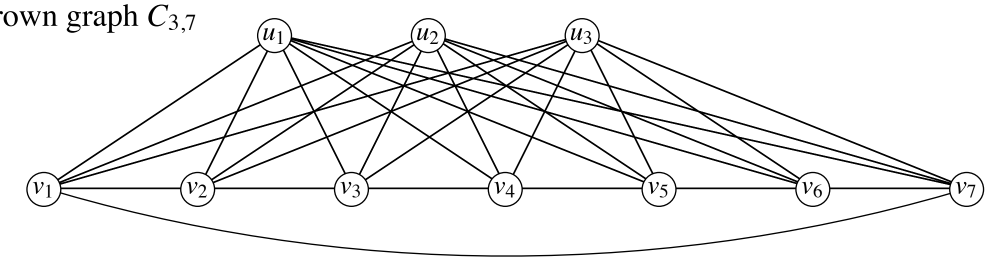
6. Footworms

A footworm grows at a constant rate of one foot per day and stops growing when it reaches one foot. A full-grown worm can be cut into two worms of lengths x and $1 - x$, each of which then grows at the constant rate until full-grown. Worms that are not full-grown cannot be cut. Show how to produce, in one day, six full-grown worms, where you start with one full-grown worm.

7. A Thorny Crown

Define the crown graph $C_{m,n}$ to consist of an n-cycle with vertices v_i, and m additional vertices u_j that are initially isolated. Then edges are added between each u_i and v_j; the diagram shows $C_{3,7}$. Show how to color the edges of $C_{7,7}$ using the smallest number of colors so that edges with a common vertex get different colors. You need only provide a complete coloring; further details on why it works or how you found it are not required.

The crown graph $C_{3,7}$



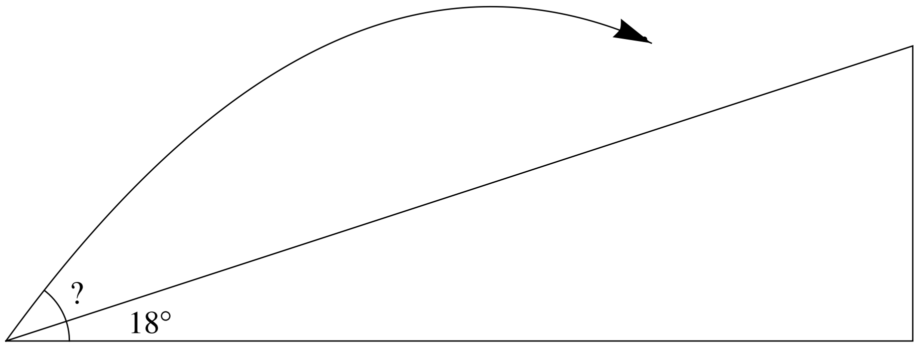
Each blank square in the table below corresponds to an edge of $C_{7,7}$. The bottom row has the cycle edges: $v_1 \bullet\!\!-\!\!\bullet v_2$, $v_2 \bullet\!\!-\!\!\bullet v_3$, and so on, up to $v_7 \bullet\!\!-\!\!\bullet v_1$ at bottom right. Fill in the squares (use the attached sheet at end of problem list) with your colors, using $1, 2, 3, \ldots$ for the colors.

	v_1	v_2	v_3	v_4	v_5	v_6	v_7
u_1							
u_2							
u_3							
u_4							
u_5							
u_6							
u_7							

cycle:							
	$v_1 \bullet\!\!-\!\!\bullet v_2$			$v_4 \bullet\!\!-\!\!\bullet v_5$			$v_7 \bullet\!\!-\!\!\bullet v_1$

8. A T-Shirt Gun

The stands at a stadium slope up at an 18° angle. You are firing a T-shirt cannon from the bottom of the stands into the stands. At what angle of elevation should you fire the cannon to get the T-shirt to land as far up the stands as possible? Ignore air resistance.



9. An Arithmetic Problem

An *arithmetic sequence* is a sequence $\{x_n\}_{n=1}^{\infty}$ with constant differences: for every m and n, the differences $x_{m+1} - x_m$ and $x_{n+1} - x_n$ are equal. Let $\{a_n\}_{n=1}^{\infty}$ be a strictly increasing arithmetic sequence of positive real numbers; for each n let $b_n = \frac{a_1+a_2+\cdots+a_n}{a_{n+1}+a_{n+2}+\cdots+a_{2n}}$.

(a) Find $\lim_{n\to\infty} b_n$.

(b) For which initial sequences $\{a_n\}$ is the sequence $\{b_n\}$ a constant sequence?

10. Where's Bob?

Agent Alice is on the trail of computer hacker Bob; she knows that he is hiding in one of 17 caves. The caves form a linear array. Bob moves only at night, and every night he moves from the cave he is in to one of the nearest caves on either side of it. Alice can search two caves each day, with no restrictions on her choice. For example, if Alice searches (1 2), (2 3), …, (16 17), then she will catch Bob, though it might take 16 days.

Find a strategy by which Alice is guaranteed to find Bob in 10 days. Partial credit will be given for strategies that require 11, 12, 13, 14, or 15 days.

Hand-in Sheet for #1

Team Code ________________________

	e	f	g	h
a				
b				
c				
d				

ACROSS

a. $\neg f$

b. $b \lor b$

c. $b \Rightarrow c$

d. $a \lor \neg h$

DOWN

e. $e \Rightarrow b$

f. $\neg c$

g. $\neg (g \Rightarrow (\neg g))$

h. $\neg \neg h$

	e	f	g	h
a				
b				
c				
d				

ACROSS

a. $\neg f$

b. $b \lor b$

c. $b \Rightarrow c$

d. $a \lor \neg h$

DOWN

e. $e \Rightarrow b$

f. $\neg c$

g. $\neg (g \Rightarrow (\neg g))$

h. $\neg \neg h$

Hand-in Sheet for #7

Team Code ________________________

	v_1	v_2	v_3	v_4	v_5	v_6	v_7
u_1							
u_2							
u_3							
u_4							
u_5							
u_6							
u_7							

cycle:							
	v_1•—•v_2			v_4•—•v_5			v_7•—•v_1

	v_1	v_2	v_3	v_4	v_5	v_6	v_7
u_1							
u_2							
u_3							
u_4							
u_5							
u_6							
u_7							

cycle:							
	v_1•—•v_2			v_4•—•v_5			v_7•—•v_1