The 2014 Michigan Autumn Take Home Challenge

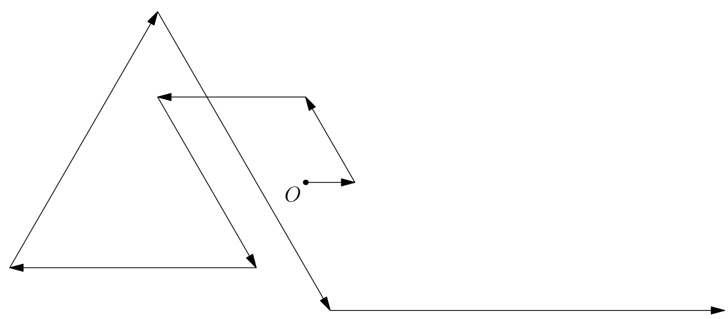
1. **Darts.** Alice and Bob are playing a game in which they take turns throwing darts at a dart board. Alice goes first. If, after any player's turn, their most recent throw is *not* the closest to the center of the dart board, they lose. Assuming that either player is uniformly likely to hit any area of the dart board on any given throw, compute the probability that Alice will win.

2. **Power fractions.** Find, with proof, the smallest possible value for $a+b$, if a and b are positive integers and the first 14 places of $\frac{1}{a}-\frac{1}{b}$ after the decimal are .01020408163265.

3. **You're a star.** Let S be a nonempty set. For any two elements $a, b \in S$, the operation $\star$ satisfies the following properties:
$$a \star (b \star a) = a, \quad \text{and} \quad (a \star b) \star b = b.$$
Show that $\star$ is idempotent; that is, show that $a \star a = a$ for all $a \in S$.

4. **Pen-tastic.** Equilateral pentagon $ABCDE$ has sides of length 1. Given
$$\cos B = \cos D = -\frac{3}{4} \quad \text{and} \quad \tan(A+C) + \tan E = 0,$$
compute the area of $ABCDE$.

5. **Voyage of the Mathbot.** A mathbot starts at the origin in the plane, and then begins a sequence of moves. On the first move, the mathbot walks 1 unit, to $(1,0)$. Between moves, the mathbot *must* rotate either 60° or 120° (either clockwise or counterclockwise). On the nth move, the mathbot walks n units. Shown below is a possible diagram of the mathbot's first 8 moves. Determine, with proof, whether the mathbot can arrive back at the origin after the 2014^{th} move.



6. **Warped integral.** The bounded continuous function f is positive and strictly increasing. If
$$\int_0^1 f\left(\sqrt{x}\right)\,\mathrm{d}x = 20$$
compute the smallest possible value for $\left\lfloor \int_0^1 f(x^2)\,\mathrm{d}x \right\rfloor$, where $\lfloor y \rfloor$ denotes the greatest integer less than or equal to y.

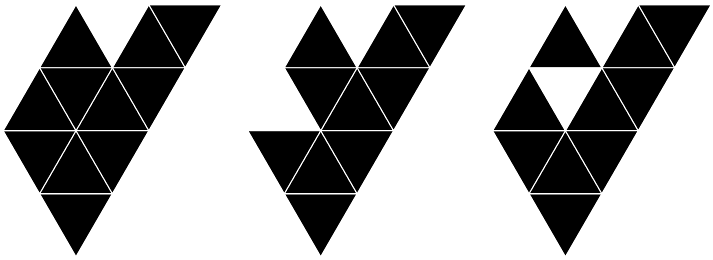
7. **New digits.** For an integer n in the domain $1 \le n \le 10^8$, define $f(n)$ to be the smallest integer greater than n that has no digits in common with the digits of n. For example, $f(2014) = 3333$ and $f(2024) = 3111$. Determine the smallest value for $x_1 + x_2 + \cdots + x_k$, if $f(x_1) + f(x_2) + \cdots + f(x_k) = 2014$.

8. **Double tangent.** Determine all real numbers c such that there exists a line ℓ passing through the origin which is tangent to the quartic polynomial $y = x^4 + 20x^3 + 14x^2 + 2014x + c$ at exactly two points.

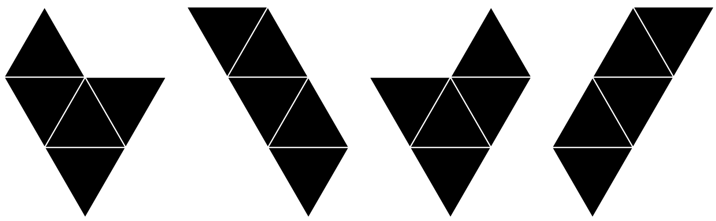
9. **Triangle-ominos.** Imagine dropping n equilateral triangles ("tiles") of side length 1, one at a time, from the positive y-axis into the region of the plane $\frac{\pi}{3} \leq \theta \leq \frac{2\pi}{3}$. Call the resulting figure a *stable triomino* if each of the tiles satisfies one of the following properties:

   - Its horizontal side is on the bottom, and directly underneath it is another tile;
   - Its horizontal side is on the top, and both the angled sides are shared with other tiles or one or both of the the boundary lines $\theta = \frac{\pi}{3}$, $\theta = \frac{2\pi}{3}$.

   In the diagram below, the first triomino is stable, but the others are not.



   Show that T_n, the number of stable triominos with n tiles, is equal to the number of ways to write n as a non-increasing sum of one or more integers congruent to $0, 1$, or $3 \bmod 4$. For example, $T_5 = 4$ – see below – and 5 can be written as $5, 4+1, 3+1+1$, and $1+1+1+1+1$.



10. **Wrong division.** In the long division problem shown below, most digits have been erased. Those that were left have all been increased or decreased by 1. (So a "5" actually represents a "4" or "6".) None of the leading digits on any line are equal to 0. Determine the original 8-digit dividend.

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                  _ _ _ _ _
_ _ _ _ ) _ _ _ _ _ _ _ _
          _ 5 _ 5
            1 _ 2 _ 5
            2 _ 8 _ _
                5 _ 8 1
                3 7 _ _
                2 _ _ 2 _
                _ _ _ _ _
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